

DEFECT FORMULA FOR NODAL COMPLETE INTERSECTION THREEFOLDS

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ABSTRACT. We generalize Werner's defect formula for nodal hypersurfaces in \mathbb{P}^4 to the case of a nodal complete intersection threefold.

1. INTRODUCTION

The main goal of this paper is to give a formula for Hodge numbers of a nodal complete intersection threefold satisfying certain non-degeneracy condition. Hodge numbers of a transversal complete intersection in a projective space can be computed from the generating function of χ_g -genus [14, Thm. 22.1.1, Thm. 22.1.2]. In the special case of a three-dimensional complete intersection X of hypersurfaces of degrees (d_1, d_2, \dots, d_r) in \mathbb{P}^{r+3} we can use the Hirzebruch–Riemann–Roch theorem for the vector bundle Ω_X^1 and the Lefschetz hyperplane theorem to compute

$$h^{1,2}(X) = \frac{1}{24}c_1c_2 - \frac{1}{2}c_3 + 1$$

and then

$$h^{1,2}(X) = \left(\frac{11}{24} \sigma_1^3 - \frac{5(r+4)}{12} \sigma_1^2 + \left(\frac{(r+4)(9r+25)}{48} - \frac{11}{12} \sigma_2 \right) \sigma_1 + \frac{5(r+4)}{12} \sigma_2 + \frac{1}{2} \sigma_3 - \frac{(3r+4)(r+4)(r+3)}{48} \right) \sigma_r + 1$$

where σ_i is the i -th elementary symmetric function evaluated at (d_1, d_2, \dots, d_r) . If $X = \{F = 0\}$ is a degree d hypersurface in \mathbb{P}^4 there is moreover isomorphism

$$H^{2,1}(X) \cong (\mathbb{k}[X_0, \dots, X_4] / \text{Jac}(F))_{2d-5}$$

of the Hodge group with degree $2d - 5$ component of the Jacobian algebra of X (an explicit isomorphism is described in [19]).

First formulae for the Hodge numbers of singular threefolds were given by Clemens [3] for double coverings of \mathbb{P}^3 branched along a nodal double surface and then by Werner [26] for nodal hypersurfaces in \mathbb{P}^4 . Clemens' and Werner's formulae relate the Hodge numbers of a resolution of a nodal double solid and a nodal hypersurface to the defect of certain linear system. These results were reproved with algebraic methods (characteristic free) and generalized to the case of hypersurfaces with A-D-E singularities satisfying certain vanishings. The proofs follow the line of [19], vanishing of a certain cohomology group breaks-up the long cohomology sequence.

Our goal is to generalize Werner's formula to the case of a nodal complete intersection in projective space, in this case the considered exact sequence does not break, instead of vanishing we explicitly describe the image of one of the maps in

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the sequence. Three dimensional node admit two types of a special resolution. The first one is the blow-up of the singular locus and is called big resolution. Small resolution replace singular point with a line, in general small resolution need not be projective. In our proofs we consider the big resolution, but the Hodge numbers of any small one follows easily.

Nodal threefolds play important role in several branches of algebraic geometry, first examples of Calabi–Yau threefolds with small absolute value of the Euler characteristic were constructed as small resolutions of nodal hypersurfaces and complete intersection (cf. [12, 14, 27, 24, 18]). A \mathbb{Q} -factorial nodal quartic 3-folds and nodal double sextic are non-rational which raised the question of minimal number of nodes on non- \mathbb{Q} -factorial nodal threefold of given type (cf. [2, 6, 17, 16, 21]). Special properties of small resolutions of nodal threefolds were used to constructed examples of Calabi–Yau spaces in positive characteristic non-liftable to characteristic zero. Contraction of a class of lines on a Calabi–Yau threefold to nodes followed by a smoothing of the nodal threefolds is the so-called conifold transition which can connect different families of Calabi–Yau threefolds ([23]).

2. PRELIMINARIES

Let $X = H_1 \cap \dots \cap H_r \subset \mathbb{P}^{r+3}$ be a nodal complete intersection in \mathbb{P}^{r+3} of smooth hypersurfaces of dimensions d_1, \dots, d_r , denote $d := d_1 + \dots + d_r$. Assume moreover that the intersections $Y = H_1 \cap \dots \cap H_{r-1}$ is smooth.

We have the following Bott-type vanishings

$$H^i(\Omega_Y^j(kX)) = 0, \quad \text{for } i + j > 4, k > 0.$$

Let $\Sigma := \text{Sing } X$ be the singular locus of X , $\mu = \#\Sigma$ – the number of nodes of X and let $\sigma : \tilde{Y} \rightarrow Y$ be the blow-up of Y at the singular locus of X . Denote by \tilde{X} the strict transform of X , let $E := \sigma^{-1}(\Sigma)$ be the exceptional divisor of σ . Then \tilde{X} is non-singular and E is a disjoint union of projective 3-spaces.

Proposition 1.

$$\begin{aligned} H^0(\Omega_Y^4(\tilde{X})) &\cong H^0(\Omega_Y^4(X)), \\ H^i(\Omega_Y^4(\tilde{X})) &= 0, \quad \text{for } i > 0, \\ H^i(\Omega_Y^4(2\tilde{X})) &\cong H^i(\Omega_Y^4(2X) \otimes \mathcal{J}_\Sigma), \quad \text{for } i \geq 0. \end{aligned}$$

Proof. We have $\Omega_Y^4(\tilde{X}) \cong \sigma^*\Omega_Y^4(X) \otimes \mathcal{O}_{\tilde{Y}}(E)$, first two assertions follows now from $\sigma_*\mathcal{O}_{\tilde{Y}}(E) \cong \mathcal{O}_Y$, $R^i\sigma_*\mathcal{O}_{\tilde{Y}}(E) = 0$, projection formula and (degenerate case) of Leray spectral sequence. Applying the direct image functor to the exact sequence $0 \rightarrow \mathcal{O}_{\tilde{Y}}(-E) \rightarrow \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_E \rightarrow 0$ we get $\sigma_*\mathcal{O}_{\tilde{Y}}(-E) \cong \mathcal{J}_\Sigma$ and $R^i\sigma_*\mathcal{O}_{\tilde{Y}}(-E) = 0$, the last assertion follows now from $\Omega_Y^4(2\tilde{X}) \cong \sigma^*\Omega_Y^4(2X) \otimes \mathcal{O}_{\tilde{Y}}(-E)$. \square

Corollary 2. *We have the following exact sequence*

$$H^0\Omega_Y^4(2X) \rightarrow H^0(\Omega_Y^4(2X) \otimes \mathcal{O}_\Sigma) \rightarrow H^1\Omega_{\tilde{X}}^3(\tilde{X}) \rightarrow 0$$

Proof. By adjunction formula $\Omega_{\tilde{X}}^3(\tilde{X}) \cong \Omega_Y^4(2\tilde{X}) \otimes \mathcal{O}_{\tilde{X}}$, assertion follows now from the previous proposition and the long exact sequence associated to

$$0 \rightarrow \Omega_Y^4(\tilde{X}) \rightarrow \Omega_Y^4(2\tilde{X}) \rightarrow \Omega_Y^4(2\tilde{X}) \otimes \mathcal{O}_{\tilde{X}} \rightarrow 0.$$

\square

Proposition 3.

$$\begin{aligned} H^i(\Omega_Y^3(\tilde{X})) &\cong H^i(\Omega_Y^3(X)), \quad i \geq 0, \\ H^i(\Omega_Y^3) &= 0, \quad i \leq 2, \end{aligned}$$

Proof. By direct computations in local coordinates we verify

$$\sigma^* \Omega_Y^3 \cong \Omega_Y^3(\log E)(-3E)$$

and so

$$\sigma^*(\Omega_Y^3(X)) \cong \Omega_Y^3(\log E)(-3E) \otimes \sigma^* \mathcal{O}_Y(X).$$

Tensoring the exact sequence

$$0 \longrightarrow \Omega_Y^3(\log E)(-E) \longrightarrow \Omega_Y^3 \longrightarrow \Omega_E^3 \longrightarrow 0$$

with $\mathcal{O}_{\tilde{Y}}(\tilde{X}) \cong \mathcal{O}_{\tilde{Y}}(-2E) \otimes \sigma^*(\mathcal{O}_Y(X))$ we get

$$0 \longrightarrow \sigma^*(\Omega_Y^3(X)) \longrightarrow \Omega_Y^3(\tilde{X}) \longrightarrow \mathcal{O}_E(-2) \longrightarrow 0.$$

Now, using the direct image operator and projection formula we get

$$\sigma_* \Omega_Y^3(\tilde{X}) \cong \Omega_Y^3(X) \quad \text{and} \quad R^i \sigma_* \Omega_Y^3(\tilde{X}) = 0,$$

the assertion follows from the Leray spectral sequence. Second assertion follows in a similar manner from the exact sequence

$$0 \longrightarrow \sigma^*(\Omega_Y^3) \otimes \mathcal{O}_{\tilde{Y}}(2E) \longrightarrow \Omega_Y^3 \longrightarrow \Omega_E^3 \longrightarrow 0$$

and the Lefschetz hyperplane theorem $H^i(\Omega_Y^3) = 0$. □

Lemma 4. *The following sequence is exact*

$$0 \longrightarrow H^1 \Omega_Y^3 \longrightarrow H^1 \Omega_Y^3(\log \tilde{X}) \longrightarrow H^1 \Omega_{\tilde{X}}^2 \longrightarrow 0$$

Proof. We have $H^0(\Omega_{\tilde{X}}^2) = H^2(\mathcal{O}_{\tilde{X}}) = 0$ ([5, Prop. 3]) and $H^2(\Omega_Y^3) = 0$ (Prop. 3), now the assertion follows by the long cohomology exact sequence derived from

$$0 \longrightarrow \Omega_Y^3 \longrightarrow \Omega_Y^3(\log \tilde{X}) \longrightarrow \Omega_{\tilde{X}}^2 \longrightarrow 0$$

□

Lemma 5. *The following sequence is exact*

$$\begin{aligned} 0 \longrightarrow H^0(\Omega_Y^3(X)) \longrightarrow H^0(\Omega_Y^3(\tilde{X})) \longrightarrow H^1(\Omega_Y^3(\log \tilde{X})) \longrightarrow \\ \longrightarrow H^1(\Omega_Y^3(X)) \longrightarrow H^1(\Omega_{\tilde{X}}^3(\tilde{X})) \end{aligned}$$

Proof. Follows from the short exact sequence

$$0 \longrightarrow \Omega_Y^3(\log \tilde{X}) \longrightarrow \Omega_Y^3(X) \longrightarrow \Omega_{\tilde{X}}^3(\tilde{X}) \longrightarrow 0$$

and previous lemmata. □

3. MAIN RESULT

Now, we shall formulate and prove our main result

Theorem 6. *Let $F_1, \dots, F_r \in S := \mathbb{k}[X_0, \dots, X_{r+3}]$ be homogeneous polynomials in $r+4$ variables such that*

- *varieties $V(F_1, \dots, F_i)$ are smooth for $i = 1, \dots, r-1$,*
- *variety $X := V(F_1, \dots, F_r)$ is a threefold with ordinary double points as the only singularities.*

Denote by $\Sigma := \text{Sing}(X)$ the set of singular points of X , $\mu := \#\Sigma$ number of its elements and $d := d_1 + \dots + d_r$. Let V be a linear combination of rows of the matrix $\bigwedge^{r-1} \text{Jac}(F_1, \dots, F_r)$ which does not vanish at any point of Σ and let I be the ideal generated by entries of V .

Then

$$h^{1,1}(\hat{X}) = 1 + \delta, \quad h^{1,2}(\hat{X}) = h^{1,2}(X_{\text{smooth}}) - \mu + \delta$$

where

$$\delta := \mu - (\dim_{\mathbb{k}} I^{2d-2r-3} - \dim_{\mathbb{k}} (I \cap \mathcal{J}_{\Sigma})^{2d-2r-3})$$

is the defect of the ideal I at the singular locus of X .

Lemma 7. *There exists an epimorphism*

$$\bigoplus_{i=1}^{r-1} S^{d+d_i-r-4} \longrightarrow H^1 \Omega_Y^3(X).$$

Proof. Let Z be a complete intersection of $r-2$ hypersurfaces H_i . Using Bertini theorem we can assume without loss of generality that $Z := H_1 \cap \dots \cap H_{r-2}$ is a smooth fivefold. By similar arguments as before we easily get exact sequences

$$\begin{aligned} H^1 \Omega_Z^4(\log Y)(X) &\longrightarrow H^1 \Omega_Y^3(X) \longrightarrow 0 \\ H^0 \Omega_Y^4(X) \otimes \mathcal{O}_Z(Y) &\longrightarrow H^1 \Omega_Z^4(\log Y)(X) \longrightarrow H^1 \Omega_Z^4(X+Y) \longrightarrow 0 \end{aligned}$$

By adjunction and the Bott vanishing we get recursively that $H^0(\Omega_Y^4(X) \otimes \mathcal{O}_Z(Y))$ is an image of $S^{d+d_{r-1}-r-4}$. Now, the lemma follows by induction. \square

Consider the following commutative diagram

$$\begin{array}{ccccccc} & & \bigoplus_{i=1}^{d_r-1} S^{d+d_i-r-4} & \xrightarrow{\alpha} & H^1(\Omega_Y^3(X)) & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \phi & & \\ H^0(\Omega_Y^4(2X)) & \xrightarrow{\delta} & H^0(\Omega_Y^4(2X) \otimes \mathcal{O}_{\Sigma}) & \xrightarrow{\gamma} & H^1(\Omega_X^3(\tilde{X})) & \longrightarrow & 0 \\ \uparrow \eta & \nearrow \theta & \uparrow \cong & & \uparrow \cong & & \\ S^{d+d_r-r-4} & & \mathbb{k}^{\mu} & & H^1(\Omega_Y^4(2X) \otimes \mathcal{J}_{\Sigma}) & & \end{array}$$

All the maps except β are determined by the proofs we presented, on the other hand the identification $H^0(\Omega_Y^4(2X) \otimes \mathcal{O}_{\Sigma}) \cong \mathbb{k}^{\mu}$ is not given explicitly.

Denote by Ω the form $\Omega := \sum_{i=0}^{r+3} X_i dX_0 \wedge \dots \wedge \widehat{dX_i} \wedge \dots \wedge dX_{r+3}$. The map θ to a function A associates Poincare residue of the form $\frac{A}{F_1 \dots F_{r-1} F_r^2} \Omega$ with respect to $\frac{dF_1}{F_1}, \frac{dF_2}{F_2}, \dots, \frac{dF_{r-1}}{F_{r-1}}$ evaluated at points of Σ . When we want to identify values of

θ with vectors we have to evaluate coefficients of resulting form, which is the same as evaluate quotients of A by $(r-1) \times (r-1)$ minors of the jacobian matrix of F_1, \dots, F_{r-1} .

At each point of Σ the jacobian matrix $\text{Jac}(F)$ has rank $r-1$, so the matrix $\bigwedge^{r-1} \text{Jac}(F)$ of $(r-1) \times (r-1)$ minors has rank 1. By our assumption all the rows of this matrix are non-zero, so at every point of Σ some columns are zero the other columns have are proportional and have only non-zero entries. It may happen however that each column vanish at some point of Σ . In order to circumvent this problem we take a random linear combination of columns (V_1, \dots, V_r) which does not vanish at any point.

Composing with α, γ, ϕ , we see that β can be identified in the same manner as θ through remaining $(r-1) \times (r-1)$ minors of the jacobian matrix $\text{Jac}(F)$ of F_1, \dots, F_r , main difference is that from S^{d+d_i-r-4} we pass through $H^0(\Omega_Y^4(X) \otimes \mathcal{O}_Z(Y))$ instead of $H^0(\Omega_Y^4(2X))$ which means that we have to multiply by F_r/F_i . Evaluating at a singular point we have to pass to the limit equal V_i/V_r . Finally, denoting $\Sigma := \{P_1, \dots, P_\mu\}$ the value of β at $A_i \in S^{d+d_i-r-4}$ is $\frac{V_i(P)A_i(P)}{V_r(P)^2}$. Denote the ideals $I = (V_1, \dots, V_r)$, $J = I \cap \mathcal{I}_\Sigma$ and by I^k (resp. J^k) vector space of degree k forms in I resp. J . We have proved the following proposition

Proposition 8.

$$\dim(\text{Im}(\delta) + \text{Im}(\beta)) = \dim I^{2d-2r-3} - \dim J^{2d-2r-3}.$$

Proof of Thm. 6. By simple linear algebra we get

$$h^1(\Omega_{\tilde{X}}^2) = h^0(\Omega_Y^4(2X)) - h^0(\Omega_Y^4(X)) - h^0(\Omega_Y^3(X)) + h^1(\Omega_Y^3(X)) - \dim(\text{Im } \beta + \text{Im } \delta).$$

Repeating the computations for a smooth complete intersection X_{smooth} of the same type we get

$$h^1(\Omega_{X_{\text{smooth}}}^2) = h^0(\Omega_Y^4(2X)) - h^0(\Omega_Y^4(X)) - h^0(\Omega_Y^3(X)) + h^1(\Omega_Y^3(X))$$

so by previous Proposition

$$h^{1,2}(\tilde{X}) = h^{1,2}(X_{\text{smooth}}) - \mu + \delta.$$

As \tilde{X} is the blow-up of μ lines in any small resolution \hat{X} we get formula for $h^{1,2}(\hat{X})$, formula for $h^{1,1}(\hat{X})$ follows now from an easy Milnor number computation. \square

4. EXAMPLES

Defect formula in main theorem can be easily implemented in a computer algebra system, we use Magma code ([1]).

Example. Denote by $X(d_1, \dots, d_r; e_1, \dots, e_{r+1})$ general complete intersection of hypersurfaces of degrees d_1, \dots, d_r in \mathbb{P}^{r+3} containing general complete intersection surface of degrees e_1, \dots, e_{r+1} . In [6] these nodal threefolds were studied as candidates for non-factorial nodal complete intersections with minimal number of nodes (cf. [16, 17]). Using our main result we check that the defect equals 1 for the following cases with $r = 2$ and $d_1 + d_2 = 6$ (Calabi–Yau cases),

d_1	d_2	e_1	e_2	e_3	μ	$h^{1,1}$	$h^{1,2}$
4	2	1	1	1	13	2	77
4	2	2	1	1	18	2	72
4	2	2	2	1	24	2	66
4	2	2	2	2	32	2	58
4	2	3	2	1	18	2	72
4	2	3	2	2	24	2	66
4	2	3	3	2	18	2	72

Example. We use our main result to verify computations of the Hodge numbers of some rigid Calabi–Yau complete intersections.

Complete intersection of four quadrics in projective space \mathbb{P}^7

$$\begin{aligned} Y_0^2 &= X_0^2 + X_1^2 + X_2^2 + X_3^2 \\ Y_1^2 &= X_0^2 - X_1^2 + X_2^2 - X_3^2 \\ Y_2^2 &= X_0^2 + X_1^2 - X_2^2 - X_3^2 \\ Y_3^2 &= X_0^2 - X_1^2 - X_2^2 + X_3^2 \end{aligned}$$

studied by van Geemen and Nygaard in [11]. Using counting points in characteristic 17 they proved that small resolution of this complete intersection is rigid, i.e. $h^{1,1} = 32, h^{1,2} = 0$. The Hodge numbers of a smooth complete intersection of four quadrics equal

$$h^{1,1} = 1, h^{1,2} = 65.$$

Using magma code we compute

$$\dim_{\mathbb{C}} I^5 = 144, \dim_{\mathbb{C}}(I \cap \mathcal{J}_{\Sigma})^5 = 79, \mu = 96, \delta = 96 - (144 - 79) = 31$$

and finally for the Hodge numbers of the van Geemen Nygaard complete intersection equals

$$h^{1,1} = 1 + 31 = 32, \quad h^{1,2} = 65 - 96 + 31 = 0.$$

as computed in [11].

For the complete intersection of a quadric in quartic in \mathbb{P}^5 given by [27]

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= x_4^2 + x_5^2 + x_6^2 \\ x_1^4 + x_2^4 + x_3^4 &= x_4^4 + x_5^4 + x_6^4 \end{aligned}$$

In this case

$$\dim_{\mathbb{C}} I^5 = 200, \dim_{\mathbb{C}}(I \cap \mathcal{J}_{\Sigma})^5 = 111, \mu = 122, \delta = 122 - (200 - 111) = 33$$

and

$$h^{1,1} = 1 + 33 = 34, \quad h^{1,2} = 89 - 122 + 33 = 0.$$

Desingularized self fiber product of the Beauville surface $\Gamma(3)$ (constructed by Schoen in [24]) is birational to the complete intersection

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 &= x_4^3 + x_5^3 + x_6^3 \\ x_1 x_2 x_3 &= x_4 x_5 x_6 \end{aligned}$$

with 108 nodes. We get

$$\dim_{\mathbb{C}} I^5 = 219, \dim_{\mathbb{C}}(I \cap \mathcal{J}_{\Sigma})^5 = 146, \mu = 108, \delta = 108 - (219 - 146) = 35$$

and

$$h^{1,1} = 1 + 35 = 36, \quad h^{1,2} = 73 - 108 + 35 = 0.$$

We have also computed Hodge numbers of nodal complete intersections studied in [18, Ch. 5] confirming Meyer's results.

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